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Boundedness in Asymmetric Oscillations*

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In this paper, we are concerned with the boundedness of all the solutions of the equation

$$x'' + ax^+ - bx^- = f(t).$$

where $f(t)$ is a smooth 2π-periodic function, a and b are positive constants $(a \neq b)$. © 1999 Academic Press

1. INTRODUCTION

Motivated by [13] and [1], we will study the boundedness of all the solutions for the following differential equation

$$x'' + ax^+ - bx^- = f(t), \tag{1}$$

where $f(t)$ is a smooth 2π-periodic function, a and b are positive constants $(a \neq b)$.

This equation was studied by Fučik [4] and Dancer [2] in their investigations of boundary value problems associated to equations with "jumping nonlinearities." For recent developments, we refer to [5], [6], [7], [16], and the reference therein.

In 1996, Ortega [13] considered the equation

$$x'' + ax^+ - bx^- = 1 + \varepsilon h(t),$$

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where the smooth function $h(t)$ is 2π-periodic in t. He proved that if $|\varepsilon|$ is sufficiently small then all the solutions are bounded, that is, if $x(t)$ is a solution, then it is defined for all $t \in \mathbf{R}$ and

$$\sup_{t \in \mathbf{R}} \left(|x(t)| + |x'(t)| \right) < +\infty.$$

This result is in contrast with the well-known phenomenon of linear resonance that occurs in the case $a = b = n^2$. For example, all the solutions of

$$x'' + n^2 x = 1 + \varepsilon \cos nt$$

are unbounded for any $\varepsilon \neq 0$.

On the other hand, when

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbf{Q}, \tag{2}$$

Alonso and Ortega [1] proved that there is a 2π-periodic function $f(t)$ such that all the solutions of (1) with large initial conditions are unbounded. Moreover, for such a $f(t)$, Eq. (1) has periodic solutions. This result also shows that the behavior of solutions of (1) is different from ones of linear equations. Indeed, the existence of periodic solutions can imply the boundedness of solutions in linear case.

From (2) it follows that there are two positive constants m and n such that

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = 2\frac{m}{n}.$$

Moreover, m and n are relatively prime.

Denote by $C(t)$ the solution of

$$x'' + ax^+ - bx^- = 0$$

with the initial condition $x(0) = 1$, $x'(0) = 0$. The derivative of $C(t)$ will be denoted by $-S(t)$. Then $C(t)$ and $S(t)$ are $[(m/n) \cdot 2\pi]$-periodic in t and $C(t) \in \mathscr{C}^2(\mathbf{R})$. Moreover, $C(t)$ is even and can be given by

$$C(t) = \begin{cases} \cos\sqrt{a}\, t, & 0 \leq |t| \leq \dfrac{\pi}{2\sqrt{a}} \\ -\sqrt{\dfrac{a}{b}} \sin\sqrt{b}\left(t - \dfrac{\pi}{2\sqrt{a}}\right), & \dfrac{\pi}{2\sqrt{a}} < |t| \leq \dfrac{m\pi}{n}. \end{cases}$$

Notice that if $a = b$, then $C(t) = \cos\sqrt{a}\, t$ and $S(t) = \sqrt{a} \sin\sqrt{a}\, t$.

For given 2π-periodic function $f(t)$, let

$$\Phi_f(\theta) = \int_0^{2\pi} f(mt) C(\theta + mt)\, dt, \qquad \theta \in \mathbf{R},$$

and

$$\mathscr{A}(f) := \left\{\theta \in \mathbf{R} : \Phi_f(\theta) = 0\right\}.$$

Then $\Phi_f(\theta)$ is a 2π-periodic function and the derivative is

$$\Phi_f'(\theta) = -\int_0^{2\pi} f(mt) S(\theta + mt)\, dt, \qquad \theta \in \mathbf{R}.$$

Indeed, we have

$$\begin{aligned}
\Phi_f(\theta) &= \int_0^{2\pi} f(mt) C(\theta + mt)\, dt = \int_{\theta/m}^{2\pi + \theta/m} f(mt - \theta) C(mt)\, dt \\
&= \int_{\theta/m}^{0} f(mt - \theta) C(mt)\, dt + \int_0^{2\pi} f(mt - \theta) C(mt)\, dt \\
&\quad + \int_{2\pi}^{2\pi + (\theta/m)} f(mt - \theta) C(mt)\, dt \\
&= \int_{\theta/m}^{0} f(mt - \theta) C(mt)\, dt + \int_0^{2\pi} f(mt - \theta) C(mt)\, dt \\
&\quad + \int_0^{\theta/m} f(ms - \theta + 2m\pi) C(ms + 2m\pi)\, ds \\
&= \int_{\theta/m}^{0} f(mt - \theta) C(mt)\, dt + \int_0^{2\pi} f(mt - \theta) C(mt)\, dt \\
&\quad + \int_0^{\theta/m} f(ms - \theta) C(ms)\, ds \\
&= \int_0^{2\pi} f(mt - \theta) C(mt)\, dt,
\end{aligned}$$

which yields that $\Phi_f(\theta)$ is 2π-periodic in θ.

The purpose of this paper is to show that if the function $\Phi_f(\theta)$ is not zero for any $\theta \in \mathbf{R}$, then every solution of (1) is bounded. More precisely, we will prove

THEOREM 1. *Suppose that $f(t) \in \mathscr{C}^6(\mathbf{S}^1)$ with $\mathbf{S}^1 = \mathbf{R}/2\pi\mathbf{Z}$ and the two different positive constants a and b satisfy*

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = 2\frac{m}{n}, \tag{3}$$

where the positive integers m and n are relatively prime. If $\mathscr{A}(f)=\emptyset$, *then all the solutions of* (1) *are bounded, i.e., if* $x(t)$ *is a solution, then it exists for all* $t\in\mathbf{R}$ *and*

$$\sup_{t\in\mathbf{R}}\left(|x(t)|+|x'(t)|\right)<+\infty.$$

COROLLARY 1. *Suppose the condition* (3) *holds and*

$$f(t)=1+\mu h(t),$$

where $h(t)\in\mathscr{C}^6(\mathbf{S}^1)$ *and* μ *is a parameter. If*

$$|\mu^{-1}|>\frac{a+b}{|a-b|}\cdot\max_{t\in\mathbf{S}^1}|h(t)|,$$

then all the solutions of (1) *are bounded.*

Proof of Corollary 1. It suffices to verify $\Phi_f(\theta)\neq 0$ for all $\theta\in\mathbf{R}$. By the periodicity and evenness of $C(t)$, we have

$$\begin{aligned}\int_0^{2\pi}C(\theta+mt)\,dt&=\int_0^{2\pi}C(mt)\,dt\\&=\frac{1}{m}\cdot\int_0^{2m\pi}C(t)\,dt=\frac{n}{m}\cdot\int_0^{2m\pi/n}C(t)\,dt\\&=\frac{n}{m}\cdot\int_{-(m\pi/n)}^{m\pi/n}C(t)\,dt=\frac{2n}{m}\int_0^{m\pi/n}C(t)\,dt\\&=\frac{2n}{m}\cdot\left(\int_0^{\pi/2\sqrt{a}}C(t)\,dt+\int_{\pi/2\sqrt{a}}^{m\pi/n}C(t)\,dt\right)\\&=\frac{2n}{m}\cdot\left(\frac{1}{\sqrt{a}}-\frac{\sqrt{a}}{b}\right)\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi}|C(\theta+mt)|\,dt&=\frac{2n}{m}\cdot\left(\int_0^{\pi/2\sqrt{a}}|C(t)|\,dt+\int_{\pi/2\sqrt{a}}^{m\pi/n}|C(t)|\,dt\right)\\&=\frac{2n}{m}\cdot\left(\frac{1}{\sqrt{a}}+\frac{\sqrt{a}}{b}\right)\\&=\frac{2n}{m}\cdot\frac{a+b}{b\sqrt{a}}.\end{aligned}$$

From the definition of $\Phi_f(\theta)$, it follows that

$$\begin{aligned}|\Phi_f(\theta)| &\geq \left|\int_0^{2\pi} C(\theta + mt)\, dt\right| - \left|\int_0^{2\pi} \mu h(t) C(\theta + mt)\, dt\right| \\ &\geq \frac{2n}{m} \cdot \frac{|b-a|}{b\sqrt{a}} - |\mu| \cdot \max_{t \in \mathbf{S}^1} |h(t)| \int_0^{2\pi} |C(\theta + mt)|\, dt \\ &> 0,\end{aligned}$$

if

$$|\mu^{-1}| > \frac{a+b}{|a-b|} \cdot \max_{t \in \mathbf{S}^1} |h(t)|.$$

This completes the proof. ∎

In [1], the authors obtained the following result:
If $\mathscr{A}(f) \neq \emptyset$ *and for any* $\theta \in \mathscr{A}(f)$

$$\Phi_f'(\theta) \neq 0.$$

Then there is an $R > 0$ *such that every solution of* (1) *with*

$$x(t_0)^2 + x'(t_0)^2 > R$$

for some $t_0 \in \mathbf{R}$ *is unbounded.*

Remark. The form of the function $\Phi_f(\theta)$ given here is slightly different from the original one in [1]. In that paper, the authors defined

$$\tilde{\Phi}_f(\theta) = \int_0^{2\pi} f(t) C^{\#}(t + \theta)\, dt, \qquad \theta \in \mathbf{R},$$

where

$$C^{\#}(t) = \frac{1}{m} \sum_{k=0}^{m-1} C(t + 2k\pi).$$

However, it is easy to see

$$\Phi_f(\theta) = \tilde{\Phi}_f(\theta).$$

In fact, from the periodicity of $C(t)$ and m and n are relatively prime, it follows that

$$C^{\#}\left(t + \frac{2\pi}{n}\right) = C^{\#}(t), \quad \forall t \in \mathbf{R},$$

which implies that $f(t)C^{\#}(t+\theta)$ is 2π-periodic in t. Hence,

$$\begin{aligned}\tilde{\Phi}_f(\theta) &= \int_0^{2\pi} f(t)C^{\#}(t+\theta)\,dt = \frac{1}{m}\int_0^{2m\pi} f(t)C^{\#}(t+\theta)\,dt\\ &= \frac{1}{m^2}\sum_{k=0}^{m-1}\int_0^{2m\pi} f(t)C(t+\theta+2k\pi)\,dt\\ &= \frac{1}{m}\sum_{k=0}^{m-1}\int_0^{2\pi} f(mt)C(mt+\theta+2k\pi)\,dt\\ &= \frac{1}{m}\sum_{k=0}^{m-1}\int_{2k\pi/m}^{2\pi+2k\pi/m} f(ms-2k\pi)C(ms+\theta)\,ds\\ &= \frac{1}{m}\sum_{k=0}^{m-1}\int_{2k\pi/m}^{2\pi+2k\pi/m} f(ms)C(ms+\theta)\,ds.\end{aligned}$$

Since the integrand in the last line is 2π-periodic in s, we have

$$\int_{2k\pi/m}^{2\pi+2k\pi/m} f(ms)C(ms+\theta)\,ds = \int_0^{2\pi} f(ms)C(ms+\theta)\,ds, \quad \forall k \in \mathbf{Z}_+$$

which yields that

$$\tilde{\Phi}_f(\theta) = \int_0^{2\pi} f(ms)C(ms+\theta)\,ds = \Phi_f(\theta).$$

The idea for proving Theorem 1 is as follows. By means of transformation theory, (1) is, outside of a large disc $\mathscr{D} = \{(x, x') \in \mathbf{R}^2 : x^2 + x'^2 \le r^2\}$ in (x, x') plane, transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is closed to a so-called twist map in $\mathbf{R}^2 \setminus \mathscr{D}$. Then a version of Moser's twist theorem [14] guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x, x') plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in \mathbf{R}^2 \times \mathbf{R}$, which confines the solutions in the interior and which leads to a bound of these solutions.

The rest of this paper is organized as follows. In Section 2, we introduce the action-angle variables (r, θ). The proof of the main result will be found in Section 3. In the last section, we give a sketch of another proof of the result in [13].

2. ACTION AND ANGLE VARIABLES

Introducing a new variable y as $x' = -y$, then Eq. (1) is equivalent to the following system:

$$x' = -y, \qquad y' = ax^{+} - bx^{-} - f(t), \tag{4}$$

which is a Hamiltonian system with the Hamiltonian function

$$H(x, y, t) = 1/2y^2 + 1/2ax^{+^2} + 1/2bx^{-^2} - f(t)x.$$

LEMMA 2.1. *For any* $(x_0, y_0) \in \mathbf{R}^2$ *and* $t_0 \in \mathbf{R}$, *the unique solution* $z(t) = (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ *of* (4) *satisfying* $z(t_0) = (x_0, y_0)$ *exists on the whole* t *axis*.

From the definition of $C(t)$ and $S(t)$, we know that $(C(t), S(t))$ is the solution of the autonomous system

$$x' = -y, \qquad y' = ax^{+} - bx^{-}$$

satisfying the initial condition $(C(0), S(0)) = (1, 0)$. Hence

$$S(t)^2 + aC^{+}(t)^2 + bC^{-}(t)^2 \equiv a.$$

Under the transformation $(r, \theta) \mapsto (x, y)$ with $r > 0$ and $\theta (\operatorname{mod} 2\pi)$, given by

$$x = \lambda r^{1/2} C\left(\frac{m\theta}{n}\right), \; y = \lambda r^{1/2} S\left(\frac{m\theta}{n}\right), \tag{5}$$

where $\lambda = \sqrt{nm^{-1}a^{-1}}$, (4) is transformed into another Hamiltonian system

$$r' = -\frac{\partial h}{\partial \theta}(r, \theta, t), \qquad \theta' = \frac{\partial h}{\partial r}(r, \theta, t), \tag{6}$$

where

$$h(r, \theta, t) = \frac{n}{m} \cdot r - 2\lambda r^{1/2} C\left(\frac{m\theta}{n}\right) f(t). \tag{7}$$

It is easy to see that h is $\mathscr{C}^6$ in r and t, $\mathscr{C}^2$ in θ.

Remark. The transformation defined by (5), similar to the corresponding one in [3], has been used in [1].

Observe that

$$r\,d\theta - h\,dt = -(h\,dt - r\,d\theta).$$

This means that if one can solve $r = r(h, t, \theta)$ from (7) as a function of h, t, and θ, then

$$\frac{dh}{d\theta} = -\frac{\partial \tau}{\partial t}(h,t,\theta), \qquad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h,t,\theta), \tag{8}$$

that is, (8) is a Hamiltonian system with Hamiltonian function $r = r(h, t, \theta)$ and now the action, angle and time variables are h, t, and θ, respectively. This trick has been used in [8].

Remark. The relation between (8) and (6) is that if $((r(t), \theta(t))$ is a solution of (6), then $(h(r(t(\theta)), \theta, t(\theta)), t(\theta))$ satisfies (8), where $t = t(\theta)$ is the inverse function of $\theta = \theta(t)$.

From (7), it follows that

$$r^{1/2} = \frac{\lambda m f(t)}{n} C\left(\frac{m\theta}{n}\right) + \frac{m g_1(h,t,\theta)}{n},$$

where

$$g_1(h,t,\theta) = \sqrt{\left(\lambda f(t) C\left(\frac{m\theta}{n}\right)\right)^2 + \frac{n}{m}h}\,.$$

Hence

$$r = \frac{m}{n}h + 2\lambda\left(\frac{m}{n}\right)^{3/2} h^{1/2} f(t) C\left(\frac{m\theta}{n}\right) + R(h,t,\theta), \tag{9}$$

where

$$R(h,t,\theta) = \frac{2m^2}{n^2}\left(\lambda f(t) C\left(\frac{m\theta}{n}\right)\right)^2 \cdot \left(1 + \frac{\lambda f(t) C\left(\frac{m\theta}{n}\right)}{g_1(h,t,\theta) + \sqrt{\frac{n}{m}h}}\right).$$

Then it is not difficult to prove that

$$\left|\frac{\partial^{k+l}}{\partial h^k\, \partial t^l} R(h,t,\theta)\right| \le c_{kl} \cdot h^{-k}, \tag{10}$$

for $k + l \le 6$ and $h \gg 1$, where c_{kl} $(k + l \le 6)$ are positive constants.

Introducing a new time variable ϑ as $\theta = n\vartheta$, then the system (8) is transformed into the form

$$\frac{dh}{d\vartheta} = -\frac{\partial}{\partial t}\mathscr{H}(h,t,\vartheta), \qquad \frac{dt}{d\vartheta} = \frac{\partial}{\partial h}\mathscr{H}(h,t,\vartheta), \tag{11}$$

where

$$\mathscr{H}(h,t,\vartheta) = mh + 2\lambda m^{3/2}n^{-1/2}h^{1/2}f(t)C(m\vartheta) + nR(h,t,n\vartheta).$$

Because n is a positive integer, the function $H(h,t,\vartheta)$ is 2π-periodic in ϑ.

3. PROOF OF THEOREM 1

This section is divided into two parts. First, we give an expression of the Poincaré map of the system (11) and then prove the statement of Theorem 1 in the Introduction.

3.1. *An Expression of the Poincaré Map of* (11)

In order to calculate the Poincaré map, we introduce a new variable v varying in the closed interval $[1/\Delta, \Delta]$ and a small positive parameter δ by the formula

$$h = \frac{1}{\delta^2}v, \qquad v \in \left[\frac{1}{\Delta}, \Delta\right],$$

where the positive constant $\Delta \in (1, +\infty)$ will be determined by (22) below. Obviously, $h \gg 1 \Leftrightarrow \delta \ll 1$.

In the new action and angle variables (v,t), the system (11) can be written in the form

$$\frac{dv}{d\vartheta} = -\frac{\partial}{\partial t}H(v,t,\vartheta,\delta), \qquad \frac{dt}{d\vartheta} = \frac{\partial}{\partial v}H(v,t,\vartheta,\delta), \tag{12}$$

where

$$H(v,t,\vartheta,\delta) = mv + 2\lambda m^{3/2}n^{-1/2}\delta C(m\vartheta)v^{1/2}f(t) + n\delta^2R(\delta^{-2}v,t,n\vartheta).$$

Moreover, from (10), it follows that the perturbation $R(\delta^{-2}v,t,n\vartheta)$ satisfies

$$\delta\left|\frac{\partial^{k+l}}{\partial v^k\,\partial t^l}R(\delta^{-2}v,t,n\vartheta)\right| \to 0 \qquad \text{as } \delta \to 0^+$$

for $k + l \le 6$.

From now on, we use the notations $o_s(1)$ and $O_s(1)$. A function $f(v,t,\vartheta,\delta)$ is said to be of order $o_s(1)$ if it is $\mathscr{C}^s$ in (v,t) and

$$\left|\frac{\partial^{k+l}}{\partial v^k\,\partial t^l}f(v,t,\vartheta,\delta)\right| \to 0 \qquad \text{as } \delta \to 0^+$$

for $k + l \le s$. Similarly, a function $f(v, t, \vartheta, \delta)$ is said to be of order $O_s(1)$ if it is $\mathscr{C}^s$ in (v, t) and for $k + l \le s$,

$$\left| \frac{\partial^{k+l}}{\partial v^k \, \partial t^l} f(v, t, \vartheta, \delta) \right| \le c_0, \qquad \text{for } \delta \ll 1$$

where the constant c_0 is independent on δ.

Now we write the system (12) explicitly:

$$\begin{cases} \dfrac{dv}{d\vartheta} = -2\delta\lambda m^{3/2} n^{-1/2} C(m\vartheta) v^{1/2} f'(t) + \delta o_5(1), \\ \dfrac{dt}{d\vartheta} = m + \delta\lambda m^{3/2} n^{-1/2} C(m\vartheta) v^{-1/2} f(t) + \delta o_5(1). \end{cases} \tag{13}$$

It is easy to show that if $\delta \ll 1$, the solution $(v(\vartheta, v_0, t_0), t(\vartheta, v_0, t_0))$ with the initial condition: $(v(0, v_0, t_0), t(0, v_0, t_0)) = (v_0, t_0)$ exists in $\vartheta \in [0, 4\pi]$ for any $(v_0, t_0) \in [1/\Delta, \Delta] \times [0, 2\pi]$. Moreover,

$$0 < \frac{1}{2\Delta} \le v(\vartheta, v_0, t_0) \le 2\Delta, \quad \forall \vartheta \in [0, 4\pi].$$

Suppose that the solution $(v(\vartheta, v_0, t_0), t(\vartheta, v_0, t_0))$ has the following expression:

$$\begin{aligned} v(\vartheta, v_0, t_0) &= v_0 + \delta F_2(\vartheta, v_0, t_0), \\ t(\vartheta, v_0, t_0) &= t_0 + m\vartheta + \delta F_1(\vartheta, v_0, t_0). \end{aligned}$$

Denote by P the Poincaré map of (12). Then

$$P(v_0, t_0) = \big(v_0 + \delta F_2(2\pi, v_0, t_0), t_0 + 2m\pi + \delta F_1(2\pi, v_0, t_0)\big).$$

From the above discussions, we know that if $\delta \ll 1$, this map is well-defined in the region $[1/\Delta, \Delta] \times [0, 2\pi]$.

If one can prove that for every $\delta \ll 1$ the map P has an invariant curve which is diffeomorphic to $v_0 = \text{const.}$, then boundedness of solutions of (1) follows from the standard arguments, (see [3], [8], [11], etc.) In order to prove the existence of such invariant curves for every $\delta \ll 1$, it suffices to verify that for every $\delta \ll 1$, the Poincaré map P satisfies all the assumptions of a variant of Moser's small twist theorem which is due to Ortega [14]. In the rest of this part, we will give an expression of $(F_1(2\pi, v_0, t_0), F_2(2\pi, v_0, t_0))$.

Since $(v(\vartheta, v_0, t_0), t(\vartheta, v_0, t_0))$ is the solution of (13), we have

$$\frac{dF_1}{d\vartheta} = \lambda m^{3/2} n^{-1/2} f(t) C(m\vartheta) \cdot (v_0 + \delta F_2)^{-1/2} + o_5(1),$$
$$\frac{dF_2}{d\vartheta} = -2\lambda m^{3/2} n^{-1/2} f'(t) C(m\vartheta) \cdot (v_0 + \delta F_2)^{1/2} + o_5(1). \quad (14)$$

As in the proof in [3], one can show that

$$F_1(\vartheta, v_0, t_0), \qquad F_2(\vartheta, v_0, t_0) = O_5(1).$$

Hence

$$v(\vartheta, v_0, t_0) = v_0 + \delta O_5(1), \qquad t(\vartheta, v_0, t_0) = t_0 + m\vartheta + \delta O_5(1).$$

From the equations of (14), it follows that

$$\begin{aligned} F_1(2\pi, v_0, t_0) &= \int_0^{2\pi} \lambda m^{3/2} n^{-(1/2)} f(t(\vartheta)) C(m\vartheta) \cdot (v_0 + \delta F_2)^{-(1/2)} d\vartheta \\ &\quad + o_5(1). \\ &= \lambda m^{3/2} n^{-(1/2)} \\ &\quad \times \int_0^{2\pi} f(t_0 + m\vartheta + \delta O_5(1)) C(m\vartheta) \cdot v_0^{-(1/2)}\, d\vartheta + o_5(1). \\ &= \lambda m^{3/2} n^{-1/2} \cdot v_0^{-(1/2)} \int_0^{2\pi} f(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta + o_5(1), \end{aligned}$$

$$\begin{aligned} F_2(2\pi, v_0, t_0) &= \int_0^{2\pi} -2\lambda m^{3/2} n^{-1/2} f'(t(\vartheta)) C(m\vartheta) \cdot (v_0 + \delta F_2)^{1/2}\, d\vartheta \\ &\quad + o_5(1) \\ &= -v_0^{1/2} \int_0^{2\pi} 2\lambda m^{3/2} n^{-1/2} f'(t(\vartheta)) C(m\vartheta)\, d\vartheta + o_5(1) \\ &= -2\lambda m^{3/2} n^{-1/2} \cdot v_0^{1/2} \\ &\quad \times \int_0^{2\pi} f'(t_0 + m\vartheta + \delta O_5(1)) C(m\vartheta)\, d\vartheta + o_5(1) \\ &= -2\lambda m^{3/2} n^{-1/2} \cdot v_0^{1/2} \int_0^{2\pi} f'(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta \\ &\quad + o_5(1). \end{aligned}$$

Now we obtain an expression of the Poincaré map P as follows:

$$P\colon \begin{cases} t_1 = t_0 + 2m\pi + \delta l_1(v_0, t_0) + \delta o_5(1), \\ v_1 = v_0 + \delta l_2(v_0, t_0) + \delta o_5(1). \end{cases} \tag{15}$$
$$(v_0, t_0) \in [1/\Delta, \Delta] \times [0, 2\pi],$$

where

$$l_1(v_0, t_0) = \lambda m^{3/2} n^{-1/2} \cdot v_0^{-(1/2)} \int_0^{2\pi} f(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta$$

and

$$l_2(v_0, t_0) = -2\lambda m^{3/2} n^{-1/2} \cdot v_0^{1/2} \int_0^{2\pi} f'(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta.$$

3.2. *The Proof of Theorem* 1

In this part, we will prove that the Poincaré P given by (15) has an invariant closed curve in $[1/\Delta, \Delta]$ for every $\delta \ll 1$. Usually, the existence of such curves is guaranteed by Moser's small twist theorem [12]. However, in the standard version, Moser's theorem is concerned with a map of the form

$$t_1 = t_0 + \kappa + \delta v_0 + \cdots,$$
$$v_1 = v_0 + \cdots,$$

where κ is a fixed number, $\delta > 0$ is a small parameter and the remaining terms (indicated by dots) are of order $\delta o_4(1)$ as $\delta \to 0$. For this reason, our map P does not meet all the conditions of Moser's theorem as $l_2 \neq 0$ and l_1 depends on t_0, it seems that one cannot apply this result directly. Fortunately, there is a variant of Moser's theorem [14] which allows us to prove the existence of invariant curves for P.

Now we state Ortega's result [14].

Let $A = [\alpha, \beta] \times \mathbf{S}^1$ be a finite cylinder with universal cover $[\alpha, \beta] \times \mathbf{R}$. Consider the map

$$\mathscr{F}\colon A \to \mathbf{R} \times \mathbf{S}^1.$$

We assume that the map has the intersection property, that is, for every Jordan curve $\Gamma \subset A$ which is homotopic to the circle $\{v_0 = \text{constant}\}$ satisfies $\mathscr{F}(\Gamma) \cap \Gamma \neq \emptyset$.

Suppose that a left of $\mathscr{F}$ has of the form

$$\begin{cases} t_1 = t_0 + 2N\pi + \delta\ell_1(v_0, t_0) + \delta\varphi_1(v_0, t_0), \\ v_1 = v_0 + \delta\ell_2(v_0, t_0) + \delta\varphi_2(v_0, t_0), \end{cases} \tag{16}$$

where N is an integer, $\delta \in (0,1)$ is a parameter and ℓ_1, ℓ_2, φ_1, and φ_2 are functions satisfying

$$\ell_1 \in \mathscr{C}^6(A), \ell_1(v_0, t_0) > 0, \ \frac{\partial \ell_1}{\partial v_0}(v_0, t_0) > 0, \quad \forall (v_0, t_0) \in A, \tag{17}$$

$$\ell_2, \varphi_1, \varphi_2 \in \mathscr{C}^5(A). \tag{18}$$

In addition we assume that there exists a function I: $A \to \mathbf{R}$ satisfying

$$I \in \mathscr{C}^6(A), \ \frac{\partial I}{\partial v_0}(v_0, t_0) > 0, \quad \forall (v_0, t_0) \in A \tag{19}$$

$$\ell_1(v_0, t_0)\frac{\partial I}{\partial t_0}(v_0, t_0) + \ell_2(v_0, t_0)\frac{\partial I}{\partial v_0}(v_0, t_0) = 0, \quad \forall (v_0, t_0) \in A. \tag{20}$$

Define the functions

$$\bar{I}(v_0) = \max_{t_0 \in \mathbf{R}} I(v_0, t_0), \ \underline{I}(v_0) = \min_{t_0 \in \mathbf{R}} I(v_0, t_0), \quad v_0 \in [a, b].$$

Since the function I is periodic in t_0, the above two functions are well-defined and finite.

LEMMA 3.1 (Theorem 3.1, [14]). *Let $\mathscr{F}$ be such that* (16)–(18) *hold. Assume in addition that there exists a function I satisfying* (19), (20), *and numbers $\tilde{\alpha}$, $\tilde{\beta}$ with*

$$\alpha < \tilde{\alpha} < \tilde{\beta} < \beta, \ \bar{I}(\alpha) < \underline{I}(\tilde{\alpha}) \le \bar{I}(\tilde{\alpha}) < \underline{I}(\tilde{\beta}) \le \bar{I}(\tilde{\beta}) < \underline{I}(\beta). \tag{21}$$

Then there exist $\varepsilon > 0$ and $\delta_0 > 0$ such that if $\delta < \delta_0$ and

$$\|\varphi_1\|_{\mathscr{C}^5(A)} + \|\varphi_2\|_{\mathscr{C}^5(A)} < \varepsilon,$$

the map $\mathscr{F}$ has an invariant curve $\Gamma \subset A$. The constant ε is independent of δ. Furthermore, if we denote by $\mu(\Gamma, \delta) \in \mathbf{S}^1$ the rotation number of $\mathscr{F}$, then

$$\lim_{\delta \to 0} \mu(\Gamma, \delta) = 0.$$

Now we go back to our map P. Since P is the Poincaré map of the Hamiltonian system (12), it is symplectic and has the intersection property in the cylinder $[1/\Delta, \Delta] \times \mathbf{S}^1$. The proof can be found in [3]. Moreover, the intersection property is preserved under a homeomorphism from the cylinder $[1/\Delta, \Delta] \times \mathbf{S}^1$ to itself.

Under the diffeomorphism

$$t = t, \qquad u = \frac{1}{v},$$

the symplectic map P, given by (15), is transformed into the following form:

$$Q\colon \begin{cases} t_1 = t_0 + 2m\pi + \delta\ell_1(u_0, t_0) + \delta o_5(1) \\ u_1 = u_0 + \delta\ell_2(u_0, t_0) + \delta o_5(1), \end{cases} \qquad (u_0, t_0) \in [1/\Delta, \Delta] \times \mathbf{S}^1,$$

where

$$\ell_1(u_0, t_0) = \lambda m^{3/2} n^{-(1/2)} \cdot u_0^{1/2} \int_0^{2\pi} f(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta,$$

$$\ell_2(u_0, t_0) = 2\lambda m^{3/2} n^{-(1/2)} \cdot u_0^{3/2} \int_0^{2\pi} f'(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta.$$

Since the functions $f(t)$ and $C(t)$ are 2π-periodic and $(m/n)\cdot 2\pi$, respectively, we have, by the assumption $\mathscr{A}(f) = \varnothing$.

$$\begin{aligned} \int_0^{2\pi} f(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta &= \int_0^{2\pi} f(m\vartheta) C(m\vartheta - t_0)\, d\vartheta \\ &= \Phi_f(-t_0) \neq 0, \quad \forall t_0 \in \mathbf{R}. \end{aligned}$$

Let

$$\mathscr{J}(t_0) = \int_0^{2\pi} f(t_0 + m\vartheta) C(m\vartheta)\, d\vartheta \in \mathscr{C}^6(\mathbf{R}).$$

Then

$$\ell_1(u_0, t_0) = \lambda m^{3/2} n^{-(1/2)} \cdot u_0^{1/2} \mathscr{J}(t_0)$$

and

$$\ell_2(u_0, t_0) = 2\lambda m^{3/2} n^{-(1/2)} \cdot u_0^{3/2} \mathscr{J}'(t_0).$$

By the condition $\mathscr{A}(f) = \varnothing$, without loss of generality, we assume

$$\mathscr{J}(t_0) > 0, \quad \forall t_0 \in \mathbf{R}.$$

Hence

$$\ell_1(u_0, t_0) > 0, \qquad \frac{\partial \ell_1}{\partial u_0}(u_0, t_0) > 0.$$

Since $\mathscr{J}(t_0 + 2\pi) = \mathscr{J}(t_0)$, one has

$$\sigma = \min_{t_0 \in \mathbf{R}} \mathscr{J}(t_0) > 0$$

and

$$\Lambda = \max_{t_0 \in \mathbf{R}} \mathscr{J}(t_0) < +\infty.$$

Moreover, $\Lambda \geq \sigma$.

Now we choose the constant Δ satisfying

$$\Delta = \left(2\frac{\Lambda}{\sigma}\right)^3 > 1. \tag{22}$$

Let

$$I(u_0, t_0) = \frac{1}{\mathscr{J}(t_0)} \cdot u_0^{1/2}.$$

It is easy to verify that

$$\ell_1(u_0, t_0)\frac{\partial I}{\partial t_0}(u_0, t_0) + \ell_2(u_0, t_0)\frac{\partial I}{\partial u_0}(u_0, t_0) = 0,$$

$$\frac{\partial I}{\partial u_0}(u_0, t_0) > 0,$$

and

$$\bar{I}(1/\Delta) < \underline{I}(\Delta_1) \leq \bar{I}(\Delta_1) < \underline{I}(\Delta_2) \leq \bar{I}(\Delta_2) < \underline{I}(\Delta),$$

where $\Delta_1 = \Delta^{-(1/3)}$ and $\Delta_2 = \Delta^{1/3}$.

We have already verified that the map Q satisfies all the conditions (17)–(21). Hence for every $\delta \ll 1$, the map Q, so the map P, has an invariant closed curve diffeomorphic to $u_0 =$ constant. The proof is completed.

Remarks. 1. From the Aubry–Mather theory [10] and the Poincaré–Birkhoff fixed point theorem, one can obtain the existence of harmonic and subharmonic solutions, as well as the existence of quasi-periodic solutions of Eq. (1). The precise statements can be found in [15] and [9].

2. When $f(t) = 1 + \varepsilon g(t)$ with $|\varepsilon|$ sufficiently small, by a similar way, one can use the standard version of Moser's small twist theorem to prove the boundedness of solutions of Eq. (1). Moreover, in this case, the condition (3) can be removed. That is, one can get another proof of Ortega's result [13]. We will give a sketch of the proof in the next section.

4. ANOTHER PROOF OF ORTEGA'S RESULT: A SKETCH

In this section, we will give a proof of the boundedness of solutions of Eq. (1) when $f(t) = 1 + \varepsilon g(t)$ with $g \in \mathscr{C}^5(\mathbf{S}^1)$.

Let

$$\omega = \frac{1}{2}\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right).$$

Then $C(t) \in \mathscr{C}^2(\mathbf{R})$ is $2\omega\pi$-periodic in t. Moreover, $C(t)$ is even and can be given by

$$C(t) = \begin{cases} \cos\sqrt{a}\,t, & 0 \le |t| \le \dfrac{\pi}{2\sqrt{a}} \\ -\sqrt{\dfrac{a}{b}}\sin\sqrt{b}\left(t - \dfrac{\pi}{2\sqrt{a}}\right), & \dfrac{\pi}{2\sqrt{a}} < |t| \le \omega\pi \end{cases}.$$

Step 1. The action-angle variables (r, θ).

Under the transformation $(r, \theta) \mapsto (x, y)$ with $r > 0$ and $\theta (\mathrm{mod}\, 2\pi)$, given by

$$x = \lambda r^{1/2} C(\omega\theta), \qquad y = \lambda r^{1/2} S(\omega\theta),$$

where $\lambda = \sqrt{\omega^{-1} a^{-1}}$ and $S = -C'$, (4) is changed into another Hamiltonian system

$$r' = -\frac{\partial h}{\partial \theta}(r, \theta, t), \qquad \theta' = \frac{\partial h}{\partial r}(r, \theta, t),$$

where

$$h(r, \theta, t) = \omega^{-1} \cdot r - 2\lambda r^{1/2} C(\omega\theta) f(t). \tag{23}$$

Step 2. Another Hamiltonian system.

From (23), it follows that

$$r(h,t,\theta) = \omega \cdot h + 2\lambda\omega^{3/2}h^{1/2}f(t)C(\omega\theta) + R(h,t,\theta),$$

where $R(h,t,\theta)$ satisfies

$$\left|\frac{\partial^{k+l}}{\partial h^k\,\partial t^l}R(h,t,\theta)\right| \le c_{kl}\cdot h^{-k},$$

for $k+l\le 5$ and $h \gg 1$, where c_{kl} $(k+l\le 5)$ are positive constants.

The new Hamiltonian system is

$$\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h,t,\theta), \qquad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h,t,\theta). \tag{24}$$

Introduce a new variable v varying in the closed interval $[1/\Delta, \Delta]$ and a small positive parameter δ by the formula

$$h = \frac{1}{\delta^2}v,$$

the positive constant $\Delta \in (1, +\infty)$. Obviously, $h \gg 1 \Leftrightarrow \delta \ll 1$.

In the new action and angle variables (v,t), the system (24) can be written in the form

$$\frac{dv}{d\theta} = -\frac{\partial}{\partial t}H(v,t,\theta,\delta), \qquad \frac{dt}{d\theta} = \frac{\partial}{\partial v}H(v,t,\theta,\delta), \tag{25}$$

where

$$H(v,t,\theta,\delta) = \omega\cdot v + 2\lambda\omega^{3/2}\delta C(\omega\theta)v^{1/2}f(t) + \delta^2 R(\delta^{-2}v,t,\theta).$$

Suppose that the solution $(v(\theta,v_0,t_0), t(\theta,v_0,t_0))$ has the following expression:

$$v(\theta,v_0,t_0) = v_0 + \delta F_2(\theta,v_0,t_0),$$
$$t(\theta,v_0,t_0) = t_0 + \omega\theta + \delta F_1(\theta,v_0,t_0).$$

Denote by P the Poincaré map of (25). Then

$$P(v_0,t_0) = \big(v_0 + \delta F_2(2\pi,v_0,t_0), t_0 + 2\omega\pi + \delta F_1(2\pi,v_0,t_0)\big).$$

From the above discussions, we know that if $\delta \ll 1$, this map is well-defined in the region $[1/\Delta, \Delta]\times[0,2\pi]$.

Step 3. An expression of the map P.

As in the proof in [3], one can show that

$$F_1(\theta, v_0, t_0), \qquad F_2(\theta, v_0, t_0) = O_4(1).$$

Hence

$$v(\theta, v_0, t_0) = v_0 + \delta O_4(1), \qquad t(\theta, v_0, t_0) = t_0 + \omega\theta + \delta O_4(1).$$

By the assumption $f(t) = 1 + \varepsilon g(t)$, it follows that

$$\begin{aligned} F_1(2\pi, v_0, t_0) &= \int_0^{2\pi} \omega^{3/2}\lambda f(t(\theta)) C(\omega\theta) \cdot (v_0 + \delta F_2)^{-(1/2)}\, d\theta + o_4(1) \\ &= \omega^{3/2}\lambda \cdot v_0^{-(1/2)} \int_0^{2\pi} C(\omega\theta)\, d\theta + \varepsilon O_4(1) + o_4(1) \\ &= 2\frac{b-a}{ab} v_0^{-(1/2)} + \varepsilon O_4(1) + o_4(1), \\ F_2(2\pi, v_0, t_0) &= -2\omega^{3/2}\lambda \int_0^{2\pi} f'(t) C(\omega\theta) \cdot (v_0 + \delta F_2)^{1/2}\, d\theta + o_4(1) \\ &= -2\omega^{3/2}\lambda \int_0^{2\pi} \varepsilon g'(t) C(\omega\theta) \cdot (v_0 + \delta F_2)^{1/2}\, d\theta + o_4(1) \\ &= \varepsilon O_4(1) + o_4(1). \end{aligned}$$

Now we obtain an expression of the Poincaré map P as follows.

$$P\colon \begin{cases} t_1 = t_0 + 2\omega\pi + 2\dfrac{b-a}{ab} \cdot \delta v_0^{-(1/2)} + \delta\big(\varepsilon O_4(1) + o_4(1)\big), \\ v_1 = v_0 + \delta\big(\varepsilon O_4(1) + o_4(1)\big), \end{cases}$$

for $(v_0, t_0) \in [1/\Delta, \Delta] \times [0, 2\pi]$. So it satisfies all the conditions of Moser's small twist theorem. ∎

Remark. In [13], $g(t)$ is assumed to be of class $\mathscr{C}^4$. However, we need $g(t) \in \mathscr{C}^5$ here. I don't know if our method can be applied to prove the boundedness of solutions in the case $g \in \mathscr{C}^4$.

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